

# A characterization of edge reflection positive partition functions of vertex coloring models

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## Abstract

Szegedy [8] showed that the partition function of any vertex coloring model is equal to the partition function of a complex edge coloring model. Using some results in geometric invariant theory, we characterize for which vertex coloring model the edge coloring model can be taken to be real valued that is, we characterize which partition functions of vertex coloring models are edge reflection positive. This answers a question posed by Szegedy [8].

## 1 Introduction

Partition functions of vertex and edge coloring models are graph invariants introduced by de la Harpe and Jones [4]. In fact, in [4] they are called spin and vertex models respectively. Both models give a rich class of graph invariants. But they do not coincide. For example the number of matchings in a graph is the partition function of a real edge coloring model but not the partition function of any real vertex coloring model. This can be deduced from the characterization of partition functions of real vertex coloring models by Freedman, Lovász and Schrijver [3]. (It is neither the partition function of any complex vertex coloring model, but we will not prove this here.) Conversely, the number of independent sets is not the partition function of any real edge coloring model, as follows from Szegedy's characterization of partition functions of real edge coloring models [8], but it is the partition function of a (real) vertex coloring model.

However, Szegedy [8] showed that the partition function of any vertex coloring model can be obtained as the partition function of a complex edge coloring model. Moreover, he gave examples when the edge coloring model can be taken to be real valued. This made him ask the question which partition functions of real vertex coloring models are partition functions of real edge coloring models (cf. [8, Question 3.2]). In fact, he phrased his question in terms of edge reflection positivity. We will get back to that in Section 3.

In this note we completely characterize for which vertex coloring models there exists a real edge coloring model such that their partition functions coincide, answering Szegedy's question.

The organization of this paper is as follows. In the next section we give definitions of partition functions of edge and vertex coloring models and state our main result (cf. Theorem 2). In Section 3 we say something about edge reflection positivity and give some applications of our result. Section 4 is devoted to proving Theorem 2.

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## 2 Partition functions of edge and vertex coloring models

We give the definitions of edge and vertex coloring models and their partition functions. After that we describe Szegedy's result how to obtain a complex edge coloring model from a vertex coloring model such that their partition functions are the same. (The existence also follows from the characterization of partition functions of complex edge coloring models given in [1], but Szegedy gives a direct way to construct the edge coloring model from the vertex coloring model.) And finally we will state our main result saying which partition functions of vertex coloring models are partition function of real edge coloring models.

Let  $\mathcal{G}$  be the set of all graphs, allowing multiple edges and loops. Let  $\mathbb{C}$  denote the set of complex numbers and let  $\mathbb{R}$  denote the set of real numbers. If  $V$  is a vector space we write  $V^*$  for its dual space, but by  $\mathbb{C}^*$  we mean  $\mathbb{C} \setminus \{0\}$ . For a matrix  $U$  we denote by  $U^*$  its conjugate transpose and by  $U^T$  its transpose.

Let  $\mathbb{F}$  be a field. An  $\mathbb{F}$ -valued *graph invariant* is a map  $p : \mathcal{G} \rightarrow \mathbb{F}$  which takes the same values on isomorphic graphs.

Throughout this paper we set  $\mathbb{N} = \{1, 2, \dots\}$  and for  $n \in \mathbb{N}$ ,  $[n]$  denotes the set  $\{1, \dots, n\}$ . We will now introduce partition functions of vertex and edge coloring models.

Let  $a \in (\mathbb{C}^*)^n$  and let  $B \in \mathbb{C}^{n \times n}$  be a symmetric matrix. Following de la Harpe and Jones [4] we call the pair  $(a, B)$  an  $n$ -color *vertex coloring model*. If moreover,  $a$  is positive and  $B$  is real, then we call  $(a, B)$  a *real  $n$ -color vertex coloring model*. When talking about a vertex coloring model, we will sometimes omit the number of colors. The *partition function* of an  $n$ -color vertex coloring model  $(a, B)$  is the graph invariant  $p_{a,B} : \mathcal{G} \rightarrow \mathbb{C}$  defined by

$$p_{a,B}(H) := \sum_{\phi: V(H) \rightarrow [n]} \prod_{v \in V(H)} a_{\phi(v)} \cdot \prod_{uv \in E(H)} B_{\phi(u), \phi(v)}, \quad (1)$$

for  $H \in \mathcal{G}$ .

We can view  $p_{a,B}$  in terms of weighted homomorphisms. Let  $G(a, B)$  be the complete graph on  $n$  vertices (including loops) with vertex weights given by  $a$  and edge weights given by  $B$ . Then  $p_{a,B}(H)$  can be viewed as counting the number of weighted homomorphisms of  $H$  into  $G(a, B)$ . In this context  $p_{a,B}$  is often denoted by  $\text{hom}(\cdot, G(a, B))$ . The vertex coloring model can also be seen as a statistical mechanics model where vertices serve as particles, edges as interactions between particles, and colors as states or energy levels.

Let for a field  $\mathbb{F}$ ,

$$R(\mathbb{F}) := \mathbb{F}[x_1, \dots, x_k] \quad (2)$$

denote the polynomial ring in  $k$  variables. We will only consider  $\mathbb{F} = \mathbb{R}$  and  $\mathbb{F} = \mathbb{C}$ . Note that there is a one-to-one correspondence between linear functions  $h : R(\mathbb{F}) \rightarrow \mathbb{F}$  and maps  $h : \mathbb{N}^k \rightarrow \mathbb{F}$ ;  $\alpha \in \mathbb{N}^k$  corresponds to the monomial  $x^\alpha := x_1^{\alpha_1} \cdots x_k^{\alpha_k} \in R(\mathbb{F})$  and the monomials form a basis for  $R(\mathbb{F})$ . Following de la Harpe and Jones [4] we call any  $h \in R(\mathbb{C})^*$  a  $k$ -color *edge coloring model*. Any  $h \in R(\mathbb{R})^*$  is called a *real  $k$ -color edge coloring model*. When talking about an edge coloring model, we will sometimes omit the number of colors. The *partition function* of a  $k$ -color edge coloring model  $h$  is the graph invariant  $p_h : \mathcal{G} \rightarrow \mathbb{C}$  defined by

$$p_h(G) = \sum_{\phi: E(G) \rightarrow [k]} \prod_{v \in V(G)} h\left(\prod_{e \in \delta(v)} x_{\phi(e)}\right), \quad (3)$$

for  $G \in \mathcal{G}$ . Here  $\delta(v)$  is the multiset of edges incident with  $v$ . Note that, by convention, a loop is counted twice.

The edge coloring model can be considered as a statistical mechanics model, where edges serve as particles, vertices as interactions between particles, and colors as states

or energy levels. Moreover, partition functions of edge coloring models generalize the number of proper line graph colorings.

We will now describe a result of Szegedy [8] (see also [9]) showing that partition functions of vertex coloring models are partition functions of edge coloring models.

Let  $(a, B)$  be an  $n$ -color vertex coloring model. As  $B$  is symmetric we can write  $B = U^T U$  for some  $k \times n$  (complex) matrix  $U$ , for some  $k$ . Let  $u_1, \dots, u_n \in \mathbb{C}^k$  be the columns of  $U$ . Define the edge coloring model  $h$  by  $h := \sum_{i=1}^n a_i \text{ev}_{u_i}$ , where for  $u \in \mathbb{C}^k$ ,  $\text{ev}_u \in R(\mathbb{C})^*$  is the linear map defined by  $p \mapsto p(u)$  for  $p \in R(\mathbb{C})$ .

**Lemma 1** (Szegedy [8]). *Let  $(a, B)$  and  $h$  be as above. Then  $p_{a,B} = p_h$ .*

Although the proof is not difficult we will give it for completeness.

*Proof.* Let  $G = (V, E) \in \mathcal{G}$ . Then  $p_h(G)$  is equal to

$$\begin{aligned}
& \sum_{\phi: E \rightarrow [k]} \prod_{v \in V} h\left(\prod_{e \in \delta(v)} x_{\phi(e)}\right) = \sum_{\phi: E \rightarrow [k]} \prod_{v \in V} \left( \sum_{i=1}^n a_i \prod_{e \in \delta(v)} u_i(\phi(e)) \right) \\
&= \sum_{\phi: E \rightarrow [k]} \sum_{\psi: V \rightarrow [n]} \prod_{v \in V} (a_{\psi(v)} \prod_{e \in \delta(v)} u_{\psi(v)}(\phi(e))) \\
&= \sum_{\psi: V \rightarrow [n]} \prod_{v \in V} a_{\psi(v)} \cdot \sum_{\phi: E \rightarrow [k]} \prod_{v \in V} \prod_{e \in \delta(v)} u_{\psi(v)}(\phi(e)) \\
&= \sum_{\psi: V \rightarrow [n]} \prod_{v \in V} a_{\psi(v)} \cdot \sum_{\phi: E \rightarrow [k]} \prod_{vw \in E} u_{\psi(v)}(\phi(vw)) u_{\psi(w)}(\phi(vw)) \\
&= \sum_{\psi: V \rightarrow [n]} \prod_{v \in V} a_{\psi(v)} \cdot \prod_{vw \in E} \sum_{i=1}^k u_{\psi(v)}(i) u_{\psi(w)}(i) = \sum_{\psi: V \rightarrow [n]} \prod_{v \in V} a_{\psi(v)} \cdot \prod_{vw \in E} B_{\psi(v), \psi(w)}.
\end{aligned} \tag{4}$$

By definition, the last line of (4) is equal to  $p_{a,B}(G)$ . This completes the proof.  $\square$

Note that the proof of Lemma 1 also shows that if  $h \in R(\mathbb{C})^*$  is given by  $h = \sum_{i=1}^n a_i \text{ev}_{u_i}$  for certain  $a \in (\mathbb{C}^*)^n$  and  $u_1, \dots, u_n \in \mathbb{C}^k$ , then  $p_h$  can be realized as the partition function of an  $n$ -color vertex coloring model. Namely take  $a = (a_1, \dots, a_n)$  and  $B = U^T U$  where  $U$  is the matrix with columns the  $u_i$ .

Let  $(a, B)$  be an  $n$ -color vertex coloring model. We say that  $i, j \in [n]$  are *twins* of  $(a, B)$  if  $i \neq j$  and the  $i$ th row of  $B$  is equal to the  $j$ th row of  $B$ . If  $(a, B)$  has no twins we call the model *twin free*. Suppose now  $i, j \in [n]$  are twins of  $(a, B)$ . If  $a_i + a_j \neq 0$ , let  $B'$  be the matrix obtained from  $B$  by removing row and column  $i$  and let  $a'$  be the vector obtained from  $a$  by setting  $a'_j := a_i + a_j$  and then removing the  $i$ th entry from it. In case  $a_i + a_j = 0$ , we remove the  $i$ th and the  $j$ th row and column from  $B$  to obtain  $B'$  and we remove the  $i$ th and the  $j$ th entry from  $a$  to obtain  $a'$ . Then  $p_{a',B'} = p_{a,B}$ . So for every vertex coloring model with twins, we can construct a vertex coloring model with fewer colors which is twin free and which has the same partition function.

We need a few more definitions to state our main result. For a  $k \times n$  matrix  $U$  we denote its columns by  $u_1, \dots, u_n$ . Let, for any  $k$ ,  $(\cdot, \cdot)$  denote the standard bilinear form on  $\mathbb{C}^k$ . We call the matrix  $U$  *nondegenerate* if the span of  $u_1, \dots, u_n$  is nondegenerate with respect to  $(\cdot, \cdot)$ . In other words, if  $\text{rk}(U^T U) = \text{rk}(U)$ . For  $l \in \mathbb{N}$ , let  $O_l(\mathbb{C})$  be the complex orthogonal group, i.e.  $O_l(\mathbb{C}) := \{g \in \mathbb{C}^{l \times l} \mid (gv, gv) = (v, v) \text{ for all } v \in \mathbb{C}^l\}$ . We think of vectors in  $\mathbb{C}^k$  as vectors in  $\mathbb{C}^l$  for any  $l \geq k$ . So in particular,  $O_l(\mathbb{C})$  acts on  $\mathbb{C}^k$ . We can now state our main result.

**Theorem 2.** *Let  $(a, B)$  be a twin-free  $n$ -color vertex coloring model. Let  $U$  be a nondegenerate  $k \times n$  matrix such that  $U^T U = B$ . Then the following are equivalent:*

- (i)  $p_{a,B} = p_y$  for some real edge coloring model  $y$ ,

(ii) there exists  $l \geq k$ ,  $g \in O_l(\mathbb{C})$  such that the set  $\left\{ \begin{pmatrix} g^{u_i} \\ a_i \end{pmatrix} \mid i = 1, \dots, n \right\}$  is closed under complex conjugation,

(iii) there exists  $l \geq k$ ,  $g \in O_l(\mathbb{C})$  such that  $\sum_{i=1}^n a_i \text{ev}_{gu_i}$  is real.

If moreover,  $UU^* \in \mathbb{R}^{k \times k}$ , then we can take  $g$  equal to the identity in (ii) and (iii).

Note that for  $h := \sum_{i=1}^n a_i \text{ev}_{gu_i}$  in the theorem above, we have, by Lemma 1,  $p_h = p_{a,B}$ . Moreover, observe that if the set of columns of  $gU$  are closed under complex conjugation, then  $gU(gU)^*$  is real. So the existence of a nondegenerate matrix  $U$  such that  $U^T U = B$  and  $UU^*$  is real, is a necessary condition for  $p_{a,B}$  to be the partition function of a real edge coloring model.

In case  $B$  is real, there is an easy way to obtain a  $k \times n$  rank  $k$  matrix  $U$ , where  $k = \text{rk}(B)$ , such that  $UU^* \in \mathbb{R}^{k \times k}$  and  $U^T U = B$ , using the spectral decomposition of  $B$ . So by Theorem 2, we get the following characterization of partition functions of real vertex colorings that are partition functions of real edge coloring models. We will state it as a corollary.

**Corollary 3.** *Let  $(a, B)$  be a twin-free real  $n$ -color vertex coloring model. Then  $p_{a,B} = p_h$  for some real edge coloring model  $h$  if and only if for each  $i \in [n]$  there exists  $j \in [n]$  such that*

(i)  $a_i = a_j$ ,

(ii) for each eigenvector  $v$  of  $B$  with eigenvalue  $\lambda$  :  $\begin{cases} \lambda > 0 & \Rightarrow v_i = v_j, \\ \lambda < 0 & \Rightarrow v_i = -v_j. \end{cases}$

### 3 Edge reflection positivity

Szegedy [8] characterized which graph invariants are partition functions of real edge coloring models in terms of multiplicativity and edge reflection positivity. To describe this characterization we need some definitions.

Let  $\mathcal{G}' := \mathcal{G} \cup \{O\}$ , where  $O$  denotes the *circle*, the graph with one edge and no vertices. Note that if  $h$  is a  $k$ -color edge coloring model, then  $p_h(O) = k$ . For any  $l \in \mathbb{N}$ , an  $l$ -fragment is a graph which has  $l$  of its vertices labeled 1 up to  $l$ , each having degree one. These labeled vertices are called the *open ends* of the fragment. An edge connected to an open end is called a *half edge*. Let  $\mathcal{F}_l$  be the collection of all  $l$ -fragments. We can identify  $\mathcal{F}_0$  with  $\mathcal{G}'$ , the collection of all graphs. Define a gluing operation  $*$  :  $\mathcal{F}_l \times \mathcal{F}_l \rightarrow \mathcal{G}'$  as follows: for  $F, H \in \mathcal{F}_l$  connect the neighbors of identically labeled open ends with an edge and then delete the open ends; the resulting graph is denoted by  $F * H$ . Note that by gluing two half edges, of which both their endpoints are open ends, one creates a circle.

For any graph invariant  $p$ , let  $M_{p,l}$  be the  $\mathcal{F}_l \times \mathcal{F}_l$  matrix defined by

$$M_{p,l}(F, H) = p(F * H), \quad (5)$$

for  $F, H \in \mathcal{F}_l$ . This matrix is called the  $l$ -th edge connection matrix of  $p$ . Graph invariants which satisfy  $p(F * G) = p(F)p(G)$  for all  $F, G \in \mathcal{F}_0$  and  $p(\emptyset) = 1$ , are called *multiplicative* and graph invariants for which the edge connection matrices are positive semidefinite are called *edge reflection positive*. We can now state Szegedy's characterization of partition functions of real edge coloring models.

**Theorem 4** (Szegedy [8]). *Let  $p : \mathcal{G}' \rightarrow \mathbb{R}$  be a graph invariant. Then there exists a real edge coloring model  $h$  such that  $p_h = p$  if and only if  $p$  is multiplicative and edge reflection positive.*

In view of Theorem 4, one could consider Corollary 3 as a characterization of those partition functions of real vertex coloring models that are edge reflection positive. It would be interesting to see whether this has any physical interpretation.

We finish this section by giving a few applications of Theorem 2.

**Example 1.** Let  $G$  be the graph on two nodes  $x_1$  and  $x_2$  with node weights equal to 1; the loop at  $x_1$  has weight 1; the loop at  $x_2$  has weight 0 and the edge  $x_1x_2$  has weight 1. Then  $\text{hom}(H, G)$  is equal to the number of independent sets of  $H$ . Using Theorem 2, it is easy to see that the partition function of any real edge coloring model can not be equal to  $\text{hom}(\cdot, G)$ . As mentioned in the introduction, this can also be easily seen using Theorem 4.

**Example 2.** For any  $n \in \mathbb{N}$  with  $n \geq 2$  consider  $K_n$ , the complete graph on  $n$  vertices. Then  $\text{hom}(H, K_n)$  is equal to the number of proper  $n$ -colorings of  $H$ . The corresponding vertex coloring model is  $(\mathbb{1}, J - I)$ , where  $\mathbb{1}$  denotes the all-ones vector,  $J$  the all-ones matrix and  $I$  the identity matrix. The eigenvalue  $-1$  of  $J - I$  has multiplicity  $n - 1$ . Using that the eigenspace corresponding to  $-1$  is equal to  $\mathbb{1}^\perp$ , it is easy to see, using Corollary 3, that  $\text{hom}(\cdot, K_n)$  is equal to the partition function of a real edge coloring model if and only if  $n = 2$ . We do not know whether it is easy to deduce this from Theorem 4. The fact that for  $n = 2$ ,  $\text{hom}(\cdot, K_n)$  is edge reflection positive was observed by Szegedy [8].

In view of Theorem 4, Example 2 shows that for each  $n \geq 3$  there exists  $k, t \in \mathbb{N}$ ,  $k$ -fragments  $F_1, \dots, F_t$  and  $\lambda \in \mathbb{R}^t$  such that  $\sum_{i,j=1}^t \lambda_i \lambda_j \text{hom}(F_i * F_j, K_n) < 0$ . It would be interesting to characterize for which (twin-free) graphs  $G$  the invariant  $\text{hom}(\cdot, G)$  is edge reflection positive. By Corollary 3, this depends on spectral properties of  $G$ .

## 4 A proof of Theorem 2

Our proof of Theorem 2 is based some on fundamental results in geometric invariant theory.

First we need some definitions and conventions. For any  $l$  and  $a \in \mathbb{C}^l$  we denote by  $\bar{a}$  the complex conjugate of  $a$ . For a square matrix  $U$ ,  $\text{tr}(U)$  denotes the *trace* of  $U$ ; the sum of the diagonal elements of  $U$ . Recall that  $O_l(\mathbb{C})$  denotes the complex orthogonal group. The real orthogonal group is the subgroup of  $O_l(\mathbb{C})$  given by all real matrices and is denoted by  $O_l(\mathbb{R})$ . Let  $k < l \in \mathbb{N}$ . We can consider any  $k$ -color edge coloring model  $h$  as an  $l$ -color edge coloring model without changing its partition function on  $\mathcal{G}$ , by setting  $h(\alpha) = 0$  if  $\alpha_i > 0$  for some  $i > k$ .

Next, we develop some framework and ideas from [1] (see also [2]). For any  $l$ , define

$$S := \mathbb{C}[y_\alpha \mid \alpha \in \mathbb{N}^l], \quad (6)$$

the polynomial ring in the infinitely many variables  $y_\alpha$ . These variables are in bijective correspondence with the monomials of  $R(\mathbb{C})$  via  $y_\alpha \leftrightarrow x_1^{\alpha_1} \cdots x_l^{\alpha_l}$ . Let  $\mathbb{N}_d^l = \{\alpha \in \mathbb{N}^l \mid |\alpha| \leq d\}$  and let  $S_d \subset S$  be the ring of polynomials in the (finitely many) variables  $y_\alpha$  with  $\alpha \in \mathbb{N}_d^l$ . Furthermore, let  $\mathcal{G}_d$  be the set of all graphs of maximum degree at most  $d$ . Let  $\mathbb{C}\mathcal{G}$  be the vector space consisting of (finite) formal  $\mathbb{C}$ -linear combinations of graphs and let  $\pi : \mathbb{C}\mathcal{G} \rightarrow S$  be the linear map defined by

$$G \mapsto \sum_{\phi: EG \rightarrow [l]} \prod_{v \in VG} y_{\phi(\delta(v))}, \quad (7)$$

for any  $G \in \mathcal{G}$ , where we consider the multiset  $\phi(\delta(v))$  as an element of  $\mathbb{N}^l$ . Note that  $\pi(G)(h) = p_h(G)$  for all  $G \in \mathcal{G}$  and  $h \in R(\mathbb{C})^*$ .

The orthogonal group acts on  $S$  via the bijection between the variables of  $S$  and the monomials of  $R(\mathbb{C})$ . Then, as was shown by Szegedy [8] (see also [1]), for any  $d$ ,

$$\pi(\mathbb{C}\mathcal{G}_d) = S_d^{O_l(\mathbb{C})}, \quad (8)$$

where  $S_d^{O_l(\mathbb{C})}$  denotes the subspace of  $S^d$  of polynomials that are  $O_l(\mathbb{C})$ -invariant. Note that the action of  $O_l(\mathbb{C})$  on  $R(\mathbb{C})$  induces an action on  $R(\mathbb{C})^*$ , i.e.  $O_l(\mathbb{C})$  acts on edge coloring models. Then (8) in particular implies that  $p_{gh} = p_h$  for all  $g \in O_l(\mathbb{C})$  and all  $h \in R(\mathbb{C})^*$ .

Now fix an  $l$ -color edge coloring model  $h$ . Let, for any  $d$ ,

$$Y_d := \{y \in \mathbb{C}^{\mathbb{N}_d^l} \mid \pi(G)(y) = p_h(G) \text{ for all } G \in \mathcal{G}_d\}. \quad (9)$$

Then  $Y_d$  is a fiber of the quotient map  $\mathbb{C}^{\mathbb{N}_d^l} \rightarrow \mathbb{C}^{\mathbb{N}_d^l} / O_l(\mathbb{C})$ . In particular,  $Y_d$  contains a unique closed orbit  $C_d$  (cf. [5, Section 8.3] or [7, Satz 3, page 101]).

Let  $\text{pr}_d : \mathbb{C}^{\mathbb{N}^l} \rightarrow \mathbb{C}^{\mathbb{N}_d^l}$  be the projection sending  $y$  to  $y_d := y|_{\mathbb{C}^{\mathbb{N}_d^l}}$ . We also write  $\text{pr}_d$  for the restriction of  $\text{pr}_d$  to  $\mathbb{C}^{\mathbb{N}_{d'}^l}$ , for any  $d' \geq d$ . Note that  $\text{pr}_d(Y_{d'}) \subseteq Y_d$  for  $d' \geq d$ , as  $\mathcal{G}_d \subseteq \mathcal{G}_{d'}$ .

The following lemma is based on results from [2].

**Lemma 5.** *Let  $h := \sum_{i=1}^n a_i e v_{u_i} \in R(\mathbb{C})^*$ , with  $a \in (\mathbb{C}^*)^n$  and distinct  $u_1, \dots, u_n \in \mathbb{C}^k$ . Suppose the bilinear form restricted to the span of the  $u_i$  is nondegenerate. If  $y$  is a real  $l$ -color edge coloring model such that  $p_h(G) = p_y(G)$  for all  $G \in \mathcal{G}$ , then there exists  $g \in O_l(\mathbb{C})$  such that  $gh = y$ .*

*Proof.* We may assume that  $l \geq k$ . Recall that in case  $l > k$  we add colors to  $h$ . This is done by appending the  $u_i$ 's with zero's. Note that the bilinear form restricted to the span of the  $u_i$  remains nondegenerate. Then, by [2, Theorem 5], for each  $d \geq 3n$ ,  $h_d \in C_d$ . Now since  $y$  is real valued, a result of Kempf and Ness [6, Theorem 0.2] implies that  $y_d \in C_d$ , for every  $d$ . We now claim that this implies that there exists  $g \in O_l(\mathbb{C})$  such that  $gh = y$ .

Indeed, define, for any  $d$ , the stabilizer of  $y_d$  by

$$\text{Stab}(y_d) := \{g \in O_l(\mathbb{C}) \mid gy_d = y_d\}. \quad (10)$$

Then  $\text{Stab}(y_d) = \cap_{d' \leq d} \text{Stab}(y_{d'})$ . Since  $O_l(\mathbb{C})$  is Noetherian there exists  $d_1 \geq 3n$  such that  $\text{Stab}(y_{d_1}) = \cap_{d \in \mathbb{N}} \text{Stab}(y_d)$ . Now since we have a canonical bijection from  $O_l(\mathbb{C}) / \text{Stab}(y_d)$  to  $C_d$ , this implies that for any  $d \geq d_1$ , if  $g \in O_l(\mathbb{C})$  is such that  $gy_d = h_d$ , then also  $gy = h$ . This proves the lemma.  $\square$

Let  $W \in \mathbb{C}^{l \times n}$  be any matrix and consider the function  $f_W : O_l(\mathbb{C}) \rightarrow \mathbb{R}$  defined by

$$g \mapsto \text{tr}(W^* g^* g W) = \text{tr}((gW)^* gW). \quad (11)$$

This function was introduced by Kempf and Ness [6] in the context of connected reductive linear algebraic groups acting on finite dimensional vector spaces. Note that  $f_W$  is left-invariant under  $O_l(\mathbb{R})$  and right-invariant under  $\text{Stab}(W) := \{g \in O_l(\mathbb{C}) \mid gW = W\}$ . Let  $e \in O_l(\mathbb{C})$  denote the identity. We are interested in the situation that the infimum of  $f_W$  over  $O_l(\mathbb{C})$  is equal to  $f_W(e)$ .

**Lemma 6.** *The function  $f_W$  has the following properties:*

- (i)  $\inf_{g \in O_l(\mathbb{C})} f_W(g) = f_W(e)$  if and only if  $WW^* \in \mathbb{R}^{l \times l}$ ,
- (ii) If  $WW^* \in \mathbb{R}^{l \times l}$ , then  $f_W(e) = f_W(g)$  if and only if  $g \in O_l(\mathbb{R}) \cdot \text{Stab}(W)$ .

*Proof.* We start by showing that

$$f_W \text{ has a critical point at } e \text{ if and only if } WW^* \in \mathbb{R}^{l \times l}. \quad (12)$$

By definition, a critical point of  $f_W$  is a point  $g$  such that  $(Df_W)_g(X) = 0$  for all  $X \in T_g(O_l(\mathbb{C}))$ , where  $T_g(O_l(\mathbb{C}))$  is the tangent space of  $O_l(\mathbb{C})$  at  $g$  and where  $(Df_W)_g$  is the

derivative of  $f_W$  at  $g$ . It is well known that the tangent space of  $O_l(\mathbb{C})$  at  $e$  is the space of skew-symmetric matrices, i.e.  $T_e(O_l(\mathbb{C})) = \{X \in \mathbb{C}^{l \times l} \mid X^T + X = 0\}$ . It is easy to see that the derivative of  $f_W$  at  $e$  is the  $\mathbb{R}$ -linear map  $(Df_W)_e \in \text{Hom}_{\mathbb{R}}(\mathbb{C}^{l \times l}, \mathbb{R})$  defined by  $Z \mapsto \text{tr}(W^*(Z + Z^*)W)$ . Now let  $Z$  be skew-symmetric and write  $Z = X + iY$ , with  $X, Y \in \mathbb{R}^{l \times l}$ . Note that  $Z$  is skew-symmetric if and only if both  $X$  and  $Y$  are skew-symmetric. Write  $W = V + iT$  with  $V, T \in \mathbb{R}^{l \times l}$ . Then  $(Df_W)_e(Z)$  is equal to

$$\begin{aligned} & \text{tr}((V^T - iT^T)(X + iY + X^T - iY^T)(V + iT)) \\ &= 2\text{tr}((V^T - iT^T)iY(V + iT)) \\ &= 2\text{tr}(T^T Y V) - 2\text{tr}(V^T Y T) = 4\text{tr}(T^T Y V), \end{aligned} \quad (13)$$

where we use that  $X$  and  $Y$  are skew symmetric, and standard properties of the trace. So  $Df_e(Z) = 0$  for all skew symmetric  $Y$  if and only if  $T^T V = V^T T$ . That is, if and only if  $WW^* \in \mathbb{R}^{l \times l}$ . This shows (12).

By a result of Kempf and Ness (cf. [6, Theorem 0.1]) we can now conclude that (i) and (ii) hold. However, we will give an independent and elementary proof.

First the proof of (i). Note that (12) immediately implies that  $f_W$  does not attain a minimum at  $e$  if  $WW^* \notin \mathbb{R}^{l \times l}$ . Conversely, suppose  $WW^* \in \mathbb{R}^{l \times l}$ . Since  $WW^*$  is real and positive semidefinite there exists a real matrix  $V$  such that  $WW^* = VV^T$ . Now note that, by the cyclic property of the trace,  $f_W(g) = \text{tr}(g^* g WW^*)$ . So we have  $f_W = f_V$ . Let  $I$  denote the identity matrix. Take any  $g = X + iY \in O_l(\mathbb{C})$ , where  $X, Y \in \mathbb{R}^{l \times l}$ . Using that  $X^T X - Y^T Y = I$ , and the fact that  $f_W$  is real valued, we find that

$$f_W(g) = \text{tr}((X^T X + Y^T Y)VV^T) = \text{tr}(VV^T) + 2\text{tr}(Y^T Y VV^T) \geq \text{tr}(VV^T) = f_W(e). \quad (14)$$

This proves (i).

Next, suppose that  $f_W(g) = f_W(e)$  for some  $g \in O_l(\mathbb{C})$ . Again, since  $WW^*$  is real and positive semidefinite there exists a real matrix  $V$  such that  $WW^* = VV^T$ . Moreover, the span of the columns of  $V$  is equal to the span of the columns of  $W$ . This implies that  $\text{Stab}(V) = \text{Stab}(W)$ . Now write  $g = X + iY$ , with  $X, Y \in \mathbb{R}^{l \times l}$ . As, by (14),  $f_W(g) = f_W(e)$  if and only if  $YV = 0$ , it follows that  $gV = XV + iYV = XV$  is a real matrix. Let  $v_1, \dots, v_n$  be the columns of  $V$ . Then, since by definition of the orthogonal group,  $(gv_i, gv_j) = (v_i, v_j)$  for all  $i, j$ , and since the  $gv_i$  are real, there exists  $g_1 \in O_l(\mathbb{R})$  such that  $g_1 g V = V$ . This implies that  $g \in O_l(\mathbb{R}) \cdot \text{Stab}(V)$ . This finishes the proof of (ii).  $\square$

The next lemma will be useful to prove Theorem 2.

**Lemma 7.** *Let  $u_1, \dots, u_n \in \mathbb{C}^k$  be distinct vectors, let  $a \in (\mathbb{C}^*)^n$  and let  $h := \sum_{i=1}^n a_i e v_{u_i}$ . Then  $h$  is a real edge coloring model if and only if the set  $\left\{ \begin{pmatrix} u_i \\ a_i \end{pmatrix} \mid i = 1, \dots, n \right\}$  is closed under complex conjugation.*

*Proof.* Suppose first that the set  $\left\{ \begin{pmatrix} u_i \\ a_i \end{pmatrix} \mid i = 1, \dots, n \right\}$  is closed under complex conjugation. Then for  $p \in R(\mathbb{R})$ ,  $h(p) = \sum_{i=1}^n a_i p(u_i) = \sum_{i=1}^n \overline{a_i p(u_i)} = \overline{h(p)}$ . Hence,  $h(p) \in \mathbb{R}$ . So  $h$  is real valued.

Now the 'only if' part. By possibly adding some vectors to  $\{u_1, \dots, u_n\}$  and extending the vector  $a$  with zero's, we may assume that  $\{u_1, \dots, u_n\}$  is closed under complex conjugation. We must show that  $u_i = \overline{u_j}$  implies  $a_i = \overline{a_j}$ . We may assume that  $u_1 = \overline{u_2}$ . Using Lagrange interpolating polynomials we find  $p \in R(\mathbb{C})$  such that  $p(u_j) = 1$  if  $j = 1, 2$  and 0 else. Let  $p' := 1/2(p + \overline{p})$ . Then  $p' \in R(\mathbb{R})$  and consequently,  $h(p') = \sum_{i=1}^n a_i p(u_i) = a_1 + a_2 \in \mathbb{R}$ . Similarly, there exists  $q \in R(\mathbb{C})$  such that  $q(u_1) = i$ ,  $q(u_2) = -i$  and  $q(u_j) = 0$  if  $j > 2$ . Setting  $q' := 1/2(q + \overline{q})$  and applying  $h$  to it, we

find that  $i(a_1 - a_2) \in \mathbb{R}$ . So we conclude that  $a_1 = \overline{a_2}$ . Continuing this way proves the lemma.  $\square$

Now we can give a proof of Theorem 2.

*Proof of Theorem 2.* Observe that since  $(a, B)$  is twin free, the columns of  $U$  are distinct. Lemma 7 implies the equivalence of (ii) and (iii) for the same  $g$  and  $l$  in (ii) and (iii). Moreover, since  $(gU)^T gU = U^T g^T gU = U^T U = B$ , for any  $g \in O_l(\mathbb{C})$ , Lemma 1 shows that (iii) implies (i).

Let  $u_1, \dots, u_n$  be the columns of  $U$  and let  $h := \sum_{i=1}^n a_i \text{ev}_{u_i}$ . We will now prove that (i) implies (iii). Let  $y$  be a real  $l$ -color edge coloring model such that  $p_{a,B} = p_y$ . Since  $U$  is nondegenerate, we may assume, by Lemma 5, that  $y = gh$  for some  $g \in O_l(\mathbb{C})$ . Now note that  $gh = \sum_{i=1}^n a_i \text{ev}_{gu_i}$ . This shows that (i) implies (iii).

Now assume that  $UU^* \in \mathbb{R}^{k \times k}$ . We will show that (i) implies (iii) with  $g = e$ . Let  $y$  be a real  $l$ -color edge coloring model such that  $p_{a,B} = p_y$ . Just as above, we may assume that  $y = \sum_{i=1}^n a_i \text{ev}_{gu_i}$  for some  $g \in O_l(\mathbb{C})$ . Lemma 7 implies that the set  $\{gu_i\}$  is closed under complex conjugation. This implies that  $gU(gU)^* \in \mathbb{R}^{l \times l}$ . So by Lemma 6 (i) the infimum of  $f_{gU}$  is attained at  $e$ . Equivalently, the infimum of  $f_U$  is attained at  $g$ . Since  $UU^* \in \mathbb{R}^{k \times k}$ , this implies, by Lemma 6 (ii), that  $g \in O_l(\mathbb{R}) \cdot \text{Stab}(U)$ . Hence  $g = g_1 \cdot s$  for some  $g_1 \in O_l(\mathbb{R})$  and  $s \in \text{Stab}(U)$ . Now note that since  $sh = h$  we have that  $h = g_1^{-1}y$  and hence  $h$  is real.  $\square$

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